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Published in:
Journal of Economic Theory

Publication date:
2002

Document Version
Peer reviewed version

[Link to publication in Tilburg University Research Portal](#)

Citation for published version (APA):
van Damme, E. E. C., & Weibull, J. (2002). Evolution in games with endogenous mistake probabilities. *Journal of Economic Theory*, 106(2), 296-315.

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Evolution in games with endogenous mistake probabilities

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July 9, 2001

Abstract.

JEL code: C72

keywords: evolution, mutation rates, mistakes

^aThe authors are grateful for helpful comments from one referee and one associate editor of this journal, as well as from Sjaak Hurkens, Jens Josephson, Alexander Matros, Arthur Robson, Maria Saez-Marti and Philippe Solal, and participants at seminars at the London School of Economics, the EEA 1999 congress in Santiago di Compostela, and the first world congress of the Game Theory Society in Bilbao, 2000. This paper is a revised and condensed version of van Damme and Weibull (1999).

Running head: Endogenous mistake probabilities

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Abstract: Bergin and Lipman (1996) show that the selection effect from the random mutations in the adaptive population dynamics in Kandori, Mailath and Rob (1993) and Young (1993) is due to restrictions on how these mutation rates vary across population states. We here model mutation rates as endogenously determined mistake probabilities, by assuming that players with some effort can control the probability of implementing the intended strategy. This is shown to corroborate the results in Kandori-Mailath-Rob (1993) and, under certain regularity conditions, those in Young (1993).

1. Introduction

It has been shown by Kandori, Mailath and Rob (1993), henceforth "KMR," and Young (1993), henceforth "Young," that adding small noise to certain adaptive dynamics in games can lead to rejection of strict Nash equilibria. Specifically, in 2×2 coordination games these dynamics allow one to conclude that in the long run the risk-dominant equilibrium (Harsanyi and Selten, 1988) will result. This surprisingly strong result has recently been challenged by Bergin and Lipman (1996) who show that it depends on specific assumptions about the mutation process, namely that the mutation rate does not vary "too much" across the different states of the adaptive process. They show that, if mutation rates at different states are not taken to zero at the same rate, then many different outcomes are possible. Indeed, any stationary state in the noise-free dynamics can be approximated by a stationary state in the noisy process, by choosing the mutation rates appropriately. In particular, any of the two strict Nash equilibria in a 2×2 coordination game may be selected in the long run.

Bergin and Lipman conclude from this lack of robustness that the nature of the mutation process must be scrutinized more carefully if one is to derive economically meaningful predictions, and they offer two suggestions for doing so, in line with the two ways of interpreting the mistakes that were suggested in the original KMR and

Young papers. Mutations may be thought of as arising from individuals' experiments or from their mistakes. In the first case, it is natural to expect the mutation rate to depend on the state - individuals may be expected to experiment less in states with higher payoffs. Also in the second case state-dependent mutation rates appear reasonable - exploring an idea proposed in Myerson (1978) one might argue that mistakes associated with larger payoff losses are less likely.

While Bergin and Lipman are right to point out that these considerations might lead to state-dependent mutation rates, they do not elaborate or formalize these ideas. Hence, it is not clear whether their concerns really matter for the conclusions drawn by KMR and Young. The aim of this study is to shed light on this issue by means of a model of mutations as mistakes. The model is based on the assumption that players "rationally choose to make mistakes" because the marginal disutility of avoiding them completely is prohibitive. It turns out that our model produces mistake probabilities that under mild regularity conditions do not vary "too much" with the state of the system. Hence, the concerns of Bergin and Lipman are irrelevant in this case.

Indeed, in the case of a symmetric 2×2 -coordination game, it is straightforward to see why such a result should come about in the KMR model. Namely, for $a > c$

and $d > b$, equilibrium $(A; A)$ is risk dominant in the game

	A	B
A	$a; a$	$b; c$
B	$c; b$	$d; d$

(1)

if and only if A is the unique best reply to the mixed strategy $\frac{1}{2}A + \frac{1}{2}B$. Hence, $(A; A)$ is risk dominant if and only if $a + b > c + d$, a condition which is equivalent to $a - c > d - b$. This latter condition amounts to saying that a mistake at $(A; A)$ involves a larger payoff loss than a mistake at $(B; B)$. Hence, any reasonable theory of endogenous mistakes should imply that mistakes at $(A; A)$ are less likely than mistakes at $(B; B)$. In other words, the basin of attraction of population state " A " should not only be "larger" than that of state " B ", it should also be "deeper" - thus making it even more difficult to upset this equilibrium. This intuition is not valid, however, in other dynamics and in asymmetric games.

The formal model we develop in this paper concerns arbitrary finite games in normal form, and is similar to the control-cost model in van Damme (1987, Chapter 4). In essence, players are assumed to have a trembling hand, and by controlling it more carefully, which involves some disutility, the amount of trembles can be reduced. Since a rational player will try harder to avoid more serious mistakes, i.e. mistakes

that lead to larger payoff losses, such mistakes will be less likely. However, they will still occur with positive probability since, by assumption, the marginal cost at low mistake probabilities exceeds the marginal (payoff) benefit. Although mistake probabilities thus depend on the associated payoff losses, and therefore also on the state of the process, we show that, under mild regularity conditions, they all go to zero at the same rate when control costs become vanishingly small in comparison with the payoffs in the game. Consequently, the results established by KMR and Young then are valid also in this model of endogenous mistake probabilities.¹

The lack of robustness noted by Bergin and Lipman has also inspired Blume (1999) and Maruta (1997). Both papers consider in symmetric 2×2 games and stochastic strategy adjustment processes with the property that, when a player has the opportunity to revise his current action, he switches from the current action to the other with probability $\sigma(x; y; \epsilon)$, where y is the current payoff, x is the payoff of the alternative action, and ϵ is a noise parameter. The papers by Blume and Maruta derive conditions on the function σ which are sufficient for the risk dominant equilibrium to be selected when the noise vanishes. Roughly speaking, it is sufficient that the switching probability depends only on the payoff difference, and is monotonic in that difference, conditions that are satisfied in our model. Hence, in the case of symmetric 2×2 games some of our analysis follow from the results in these papers. Note, however, that our model concerns arbitrary finite games and that our focus

here is on the asymmetric case. Moreover, Blume and Maruta take random choice behavior as a starting point for the analysis, while we here derive such behavior from an explicit decision-theoretic model in which individuals take account of their own mistake probabilities. Hence, our work is complementary to theirs. Our paper is also related to Robles (1998), in which the KMR and Young models are extended by letting mutation rates decline to zero over time, in one part of the study also allowing for state-dependent mutation rates. Also this work is complementary in the sense that while Robles takes the state-dependence for given and analyzes implications thereof, we suggest a model that explains why and how mutation rates vary across population states.

The remainder of the paper is organized as follows. In Section 2 we introduce games with endogenous mistake probabilities. In Section 3 we introduce two regularity conditions that together guarantee that any ratio of mistake probabilities is bounded away from zero as the weight attached to the disutility of mistake control is taken to zero. In Section 4 we consider the adaptive dynamics in Young (1993, 1998) with mutations as endogenously determined mistakes. We show that, if our regularity conditions are satisfied, then Young's results for finite n -player games hold, implying, in particular, that the risk dominant equilibrium is selected in generic 2×2 coordination games. Section 5 provides two counter-examples, each example violating one of the two regularity conditions. In one of these examples, we show that there exists

no unique stochastically stable equilibrium (unlike in Young's original model), but that the risk dominant equilibrium nevertheless belongs to the limit set. Section 6 concludes. Some proofs have been relegated to an appendix at the end of the paper.

2. Endogenous mistake probabilities

Van Damme (1987, chapter 4) develops a model where mistakes arise in implementing pure strategies in games. The basic idea is that players make mistakes because it is too costly to prevent these completely. Each player has a trembling hand, and by making effort to control it more carefully, which involves disutility, the amount of trembles can be reduced. It is assumed that the disutility of eliminating trembles completely is prohibitive.

We here elaborate a closely related model of mistake control. Consider a finite n -player normal-form game G , with pure strategy sets S_1, \dots, S_n , and mixed-strategy sets $\Phi_i = \Phi(S_i)$. For any mixed-strategy profile $\mathbf{x} = (x_1, \dots, x_n)$, let the payoff to player i be $u_i(\mathbf{x}) \in \mathbb{R}$, and let $\sigma_i(\mathbf{x}) \subseteq \Phi_i$ be i 's set of best replies to $\mathbf{x} \in \Phi = \prod_{i=1}^n \Phi_i$. Let V denote the class of functions $v : (0, 1] \rightarrow \mathbb{R}_{++}$ which are twice differentiable with $v'(\epsilon) < 0$ and $v''(\epsilon) > 0$ for all $\epsilon \in (0, 1)$, $\lim_{\epsilon \rightarrow 0} v'(\epsilon) = -\infty$, and $v'(1) = 0$. Such a function v will be called a (mistake-control) disutility function.

Embed a game G as described above in an n -player game $\mathcal{G}(\epsilon; \mathbf{v}; \pm)$, for $\mathbf{v} = (v_1, \dots, v_n) \in V^n$, $\epsilon \in \text{int}(\Phi)$, and $\pm > 0$, with strategy sets $X_i = \Phi_i \times (0, 1]$, $X = \prod_{i=1}^n X_i$,

and payoff functions $u_i : X \rightarrow \mathbb{R}$ defined by

$$u_i(x) = \pi_i(\sigma_i) + \alpha v_i(\mu_i), \quad (2)$$

where $x = (x_1, \dots, x_n)$, $x_i = (\sigma_i, \mu_i)$ for all i , and

$$\pi_i = (1 - \mu_i) \pi_i + \mu_i \pi_i^*. \quad (3)$$

Our interpretation is that each player i chooses a pair $x_i = (\sigma_i, \mu_i)$, where σ_i is a mixed strategy in G and μ_i is a mistake probability. Given this choice, strategy σ_i is implemented with probability $1 - \mu_i$, otherwise the (exogenous) error distribution π_i^* is implemented. These random draws are statistically independent across player positions. Associated with each mistake-probability level μ_i is a disutility $v_i(\mu_i)$ to player i , from the effort to keep his or her mistake probability at μ_i . The "disutility weight" α measures the importance of this disutility of control effort relative to the payoffs in the underlying game G .

The assumptions made above concerning the disutility functions v_i imply that the marginal disutility of reducing one's mistake probability is increasing as the probability goes down, and that the marginal disutility of reducing it to zero is prohibitive. The associated strategy profile $\sigma \in \prod_{i=1}^n (\Sigma_i)$, defined in equation (3), is the profile that will be played in G when the players choose strategies $x_i = (\sigma_i, \mu_i)$ in $\mathcal{G}(\Sigma; \Psi; \alpha)$.

We will say that the profile x in $\mathbb{G}(\gamma; \psi; \pm)$ induces the profile $!$ in G , and we call $\mathbb{G}(\gamma; \psi; \pm)$ a game with endogenous mistake probabilities. The limiting case $\pm = 0$ represents a situation in which all players are fully rational in the sense of being able to perfectly control their actions at no effort - as if they played game G .

It is not difficult to characterize Nash equilibrium in $\mathbb{G}(\gamma; \psi; \pm)$. First, it follows directly from equations (2) and (3) that a necessary condition is $\beta_i \in \pi_i(!)$ for every player i who chooses $\alpha_i < 1$, while for players i with $\alpha_i = 1$ any $\beta_i \in \Phi_i$ is optimal. To see this, note that

$$u_i(x) = (1 - \alpha_i) \beta_i(\beta_i; !_{-i}) + \alpha_i \beta_i(\gamma; !_{-i}) - \beta_i(\alpha_i). \quad (4)$$

In other words, irrespective of the chosen mistake probability α_i , as long as this is less than one, player i will choose a best reply β_i in G to the induced profile $!$.² Thus, $\beta_i(\beta_i; !_{-i}) = b_i(!)$, the best payoff that player i can obtain in G against a strategy profile $! \in \Phi$, where

$$b_i(!) = \max_{\beta_i} \beta_i(\beta_i; !_{-i}). \quad (5)$$

Let $l_i(!) \geq 0$ denote the payoff loss that player i incurs in G if his error-distribution

$\hat{\sigma}_i$ is played against $! \in \Sigma$:

$$l_i(!) = b_i(!) - \lambda_i(\hat{\sigma}_i; !, i). \quad (6)$$

It follows from equation (4) that, in Nash equilibrium, each σ_i necessarily is the unique solution to the first-order condition

$$- \lambda_i^0(\sigma_i) = l_i(!) = \pm, \quad (7)$$

Note that $\sigma_i = 1$ if and only if $\hat{\sigma}_i \in \Sigma_i(!)$.³ Equation (7) simply says that, unless $\hat{\sigma}_i$ happens to be a best reply to $!$, the mistake probability should be chosen such that the marginal disutility of a reduction of the mistake probability equal the payoff loss in case of a mistake. If $\hat{\sigma}_i$ is a best reply to $!$, then no effort to reduce the mistake probability is worthwhile. In sum: if $x = (x_1; \dots; x_n)$, with $x_i = (\lambda_i; \sigma_i)$ for all i , is a Nash equilibrium of $G(\hat{\sigma}; \Psi; \pm)$, then each σ_i satisfies equation (7), and $\lambda_i \in \Sigma_i(!)$ if $\sigma_i < 1$. Conversely, if each σ_i satisfies equation (7), and if $\lambda_i \in \Sigma_i(!)$ if $\sigma_i < 1$, then $x = (x_1; \dots; x_n)$, for $x_i = (\lambda_i; \sigma_i)$, is a Nash equilibrium of $G(\hat{\sigma}; \Psi; \pm)$.

It follows from the assumed properties of the marginal-utility function $-\lambda_i^0$ that it has an inverse, which we denote $f_i : \mathbb{R}_+ \rightarrow (0; 1]$. Clearly f_i is differentiable with $f_i' < 0$, $f_i(0) = 1$, and $\lim_{y \rightarrow 1} f_i(y) = 0$. Equation (7) can be re-written in terms of

this inverse function as $\sigma_i = \sigma_i(\pm; !)$, where

$$\sigma_i(\pm; !) = f_i [l_i(!) = \pm] . \quad (8)$$

If the disutility weight \pm is reduced, then each player i chooses a smaller mistake probability against every profile $!$ to which σ_i is not a best reply (recall that σ_i is a best reply to $!$ if and only if all pure strategies available to the player are best replies to $!$):

$$\pm^0 < \pm \text{ and } \sigma_i \not\geq \sigma_i(!) \quad) \quad \sigma_i(\pm^0; !) < \sigma_i(\pm; !) . \quad (9)$$

In particular, if $\sigma_i \not\geq \sigma_i(!)$ then $\sigma_i(\pm; !) \rightarrow 0$ as $\pm \rightarrow 0$. In other words, in the limit case of fully rational players (zero cost of mistake control), no mistakes are made unless all pure strategies happen to be best replies.

Moreover, (8) implies that a player chooses a smaller mistake probability if the expected loss in case of a mistake is larger:

$$l_i(!) > l_i(!^0) \quad) \quad \sigma_i(\pm; !) < \sigma_i(\pm; !^0) . \quad (10)$$

3. Regular disutility function profiles

There are two questions that are relevant concerning the order of magnitude of mistake probabilities when the disutility weight \pm is taken to zero:

- i) Will the mistake probabilities of one player be of the same order of magnitude as the mistake probabilities of another player, at any given strategy profile?
- ii) Will the mistake probabilities of one player be of the same order of magnitude at different strategy profiles?

3.1. Similarity. Our formulation assumes that different players assign the same weight \pm to the disutility of control effort. Hence, already this assumption introduces some comparability between players. Such comparability makes sense, for example, if individuals are drawn from the same background population, in which case one might even assume that all player populations have the same disutility function. However, the latter is not needed for our results. It is sufficient that the disutility functions of different player populations are similar in the sense that

$$\lim_{y \downarrow 1} \inf_i f_i(y) = f_j(y) > 0 \quad \forall i, j \in I, \quad (11)$$

(where f_i is the inverse of the marginal-utility function u_i^0 , see section 2). This condition is for example met if one player's disutility function is proportional to

another player's disutility function. A more general sufficient condition for similarity is that there for every pair $(i; j)$ of individuals exist a continuously differentiable function $g_{ij} : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ such that $v_j(\alpha) = g_{ij}[v_i(\alpha)]$ for all $\alpha \in (0; 1]$, where $a_{ij} < g_{ij}^0 < b_{ij}$, for some positive real numbers a_{ij} and b_{ij} .

In section 5 we show that, without a similarity assumption of this kind, the results for adaptive population processes in Young (1993,1998) need not hold.

3.2. Niceness. We now turn to the issue of whether mistake probabilities of the same player, associated with different strategy profiles, μ and μ^0 , will be of the same order in the limit. In the appendix, we show that the mere existence of this limit, together with a natural additional condition, is sufficient. To be more specific, we show that the limit superior of the ratio of any two error probabilities is positive. Unfortunately, as shown in section 5, one can construct artificial examples in which the ratio $\mu_i(\pm; \mu) / \mu_i(\pm; \mu^0)$ between a player's mistake probabilities at two strategy profiles, μ and μ^0 , does not converge as \pm is taken to zero, and where the limit inferior is zero. We will show that such examples are excluded by the following condition on the disutility functions:

Definition 1. $v_i \in V$ is nice if $\liminf_{y \downarrow 1} f_i(\alpha y) / f_i(y) > 0$ for some $\alpha > 1$.

It turns out that under a mild additional condition, the corresponding limit superior is positive. The condition is that the disutility, not only the marginal disutility, of

bringing down the mistake probability to zero is prohibitive.

Proposition 1. Suppose $\lim_{\mu \rightarrow 0} v_i(\mu) = +\infty$. Then

$$\limsup_{\mu \rightarrow 1} f_i(\mu, y) = f_i(y) > 0 \quad \text{for all } \mu > 0$$

(A proof is given in the appendix.) If the limit of $f_i(\mu, y) = f_i(y)$ as $\mu \rightarrow 1$ exists for some $\mu > 0$, then the limit inferior coincides with this limit, and hence this limit is positive under the hypothesis of the lemma. It follows that v_i is nice whenever the limit exists. In this sense, "niceness" is indeed a weak requirement. Note also that definition 1 only requires the limit inferior to be positive for some $\mu > 0$. It is, however, easily verified that if v_i is nice, then the limit inferior in the definition actually holds for all $\mu > 0$ (see appendix for a proof).

Niceness is defined rather indirectly, in terms of the inverse to the derivative of the disutility function in question. The next result (of which the proof is again in the appendix) provides a sufficient condition for niceness directly in terms of the disutility function. It essentially says that the (positive) second derivative of the disutility function should not be increasing with μ , and that the relative risk aversion of the disutility function should not tend to zero at small mistake probabilities.⁴

Lemma 1. $v_i \in V$ is nice if it is trice differentiable with $v_i^{(0)} \geq 0$ and

$$\liminf_{\epsilon \rightarrow 0} [\epsilon v_i^{(0)}(\sigma) - v_i^{(0)}(\sigma)] > 0:$$

An example of a class of nice disutility functions is $v_i(\sigma) = 1 - \sigma_i^\alpha (1 - \sigma_i)$, where $\alpha \in (0, 1)$. In this case, $v_i^{(0)}(\sigma) = \alpha (1 - \sigma_i^{\alpha-1}) < 0$, $v_i^{(1)}(\sigma) = \alpha (1 - \sigma_i^\alpha) \sigma_i^{\alpha-2} > 0$, and $v_i^{(2)}(\sigma) = \alpha (1 - \sigma_i^\alpha) (\alpha - 2) \sigma_i^{\alpha-3} < 0$. Consequently, $\epsilon v_i^{(0)}(\sigma) - v_i^{(0)}(\sigma) = (1 - \sigma_i^\alpha) = (1 - \sigma_i^{\alpha-1}) (1 + \sigma_i^\alpha) > 0$ as $\sigma_i \neq 0$. $f_i(y) = 1/(1 + y)$. Moreover, $f_i(y) = (1 + y^\alpha)^{\frac{1}{\alpha-1}}$, so $f_i(\sigma + y) - f_i(\sigma) = \frac{\sigma + y}{\sigma + y} \frac{1}{\sigma^{\frac{1}{\alpha-1}}} - \frac{1}{\sigma^{\frac{1}{\alpha-1}}} > 0$ as $y \neq 1$, for every $\sigma > 0$. Another example of a nice disutility function is the logarithmic case, $v_i(\sigma) = -\sigma_i \ln \sigma_i$, as the reader easily verifies.

3.3. Regularity. We are now in a position to state the advertised result. If the disutility function profile \mathbf{v} is regular in the sense that all disutility functions are nice and pair-wise similar, then all mistake probabilities are of the same order of magnitude at all strategy profiles where all players have positive losses:

Proposition 2. Consider a sequence of games $G(\cdot; \mathbf{v}; \epsilon_t)$ with \mathbf{v} regular, and $\epsilon_t \rightarrow 0$. Suppose $\sigma_i \in \mathbb{R}^n$, and $I_j(\sigma) > 0$. Then

$$\liminf_{t \rightarrow \infty} \frac{v_i(\sigma_t; \epsilon_t)}{v_j(\sigma_t; \epsilon_t)} > 0.$$

Proof: If $l_i(\epsilon) = l_j(\epsilon^0)$, then $\mu_i(\pm; \epsilon) \leq \mu_j(\pm; \epsilon^0)$ for all t , and thus the limit inferior is at least 1. If $l_i(\epsilon) > l_j(\epsilon^0) > 0$, then write $\epsilon = l_i(\epsilon) = l_j(\epsilon^0) > 1$, and we have

$$\begin{aligned} \liminf_{\epsilon \rightarrow 0} \frac{\mu_i(\pm; \epsilon)}{\mu_j(\pm; \epsilon^0)} &= \liminf_{\epsilon \rightarrow 0} \frac{f_i[l_i(\epsilon) = \pm]}{f_j[l_j(\epsilon^0) = \pm]} = \lim_{y \rightarrow 1} \inf_{y \geq 1} \frac{f_i(\epsilon y)}{f_j(y)} \\ &= \lim_{y \rightarrow 1} \inf_{y \geq 1} \frac{f_i(y) f_i(\epsilon y)}{f_j(y) f_i(y)} > 0, \end{aligned} \quad (12)$$

where the inequality follows from niceness and similarity: $f_i(\epsilon y) = f_i(y) > a$ and $f_i(y) = f_j(y) > b$ for some $a, b > 0$, for all y sufficiently large. End of proof.

We conclude by noting two simple but relatively stringent conditions under which all mistake probabilities necessarily are of the same order of magnitude, for all players and mixed-strategy profiles. The first condition is that both the limit inferior and the limit superior of $\mu_i^0(\epsilon)$, as $\epsilon \rightarrow 0$ are positive real numbers, for all player positions i . This condition requires all disutility functions to be more or less logarithmic at small mistake probabilities. The second sufficient condition is that $\lim_{\epsilon \rightarrow 0} \mu_i^0(\epsilon) = +1$ for all player positions i . For proofs of these assertions, we refer to van Damme (1983, Theorems 4.4.1 and 4.4.2). There, a slightly different model of mistake control is studied, the main conclusion being that, under certain regularity conditions, mistake probabilities are of the same order, and, hence, that the assumption underlying Myerson's properness concept cannot be justified in this way.

4. Boundedly rational adaptation

Kandori, Mailath and Rob (1993) analyze situations where a symmetric two-player game is recurrently played by a population of individuals. In each period, all pairs of individuals play against each other, and all individuals play pure strategies. Each individual plays a best reply to last period's strategy distribution with probability $1 - \mu$, for $\mu > 0$ small. With the remaining probability, the individual plays a suboptimal strategy, with equal probability for all suboptimal strategies, and with statistical independence across individuals and periods. Hence, for any positive such mutation rate μ , this defines an ergodic Markov process. Kandori, Mailath and Rob (1993) study the limit as $\mu \rightarrow 0$ of the associated unique stationary strategy distribution. In the case of a symmetric 2×2 -coordination game, they show that this limiting distribution places all probability mass on the risk-dominant equilibrium.

It is easy to see that this result holds also under the present model of endogenous mistake probabilities mistakes. In fact, this is true without any regularity conditions imposed on the disutility function; it only relies on the monotonicity property (10).⁵ The equilibrium $(A; A)$ is risk dominant if and only if a mistake at $(A; A)$ involves a larger payoff loss than a mistake at $(B; B)$. Hence, if mutation rates are modelled as endogenously determined mistake probabilities, then mistakes in state $\backslash A$ are less likely than mistakes in state $\backslash B$. In other words, the basin of attraction of state $\backslash A$

is not only "larger" than that of state "B", it is also "deeper" - thus making state "A" even more difficult to upset. (See van Damme and Weibull (1999) for details).

This simple intuition is not available in asymmetric games. In the risk-dominant equilibrium, one of the two deviation losses may be quite small, thus inducing relatively large mistake probabilities in that player position, as is shown in a counterexample in section 5. However, under the regularity conditions introduced in section 3 above, all mistake probabilities will be of the same order of magnitude, and below we show that this implies that the results from Young (1993, 1998) carry over to this case.

4.1. Young's model. Let an n -person game G , as defined above, be given and, for each player position $i = 1; \dots; n$, let C_i be a finite population of individuals. In the Young model, the game is played recurrently between individuals, one from each population, who are randomly drawn from these populations. The state h of the system in his model is a full description of the pure-strategy profiles played in the last m such rounds ("the recent m -history"). Hence, $h \in H = S^m$. Each individual drawn to play in position i of the game is assumed to make a statistically independent sample of k of these m profiles, and plays a best reply to the opponent population's empirical frequency of actions (pure strategies) in the sample.

Noise is added to this selection process. Basically, an individual in player position

i plays a best response with probability $1 - \mu_i$. In case of multiple best replies, all are played with positive probability. A mutation occurs with positive probability μ_i , with statistical independence across time, states and individuals. Hence, all mutation rates μ_i are positive, and the ratios between mutation rates across states and player positions are constant as $\mu_i \neq 0$. Once a mutation occurs, $q_i(t, j, h) \in \text{int}[\Phi_i]$ is the conditional error distribution over player i 's pure-strategy set.

With these mistakes as part of the process, each state of the system is reachable with positive probability from every other state. Hence, the full process is an irreducible and aperiodic Markov chain on the finite state space H . Consequently, the process is ergodic and has a unique invariant distribution π^m for each $m > 0$. Young establishes the existence of the limit distribution $\pi^\infty = \lim_{m \rightarrow \infty} \pi^m$, and studies its properties. He calls an equilibrium of the underlying game G stochastically stable if π^∞ places positive probability weight on the state in which this equilibrium is played (in the last m periods).

One noteworthy result is that, for a 2×2 game with a unique risk-dominant equilibrium, this is also the unique stochastically stable equilibrium. More generally, Young (1998) establishes that, for a generic class of finite n -player games, the limit distribution places all probability mass in one of the game's minimal curb sets, more exactly on a minimal curb set where the stochastic potential (in Young's model) is minimized (Theorem 7.2 in Young (1998)).⁶ In the case of a 2×2 coordination game,

there are two minimal curb sets, each corresponding to one of the strict equilibria, and such an equilibrium is risk dominant if and only if its stochastic potential is lower than that of the other equilibrium.

4.2. The Bergin-Lipman critique. Young derives his results under the assumption that the ratio between any pair of mutation probabilities is kept constant as $\epsilon \rightarrow 0$. Bergin and Lipman (1996) note that the results continue to hold even with state-dependent mutation probabilities if the ratio between any pair of mutation probabilities, across all population states and player positions, has a positive limit when $\epsilon \rightarrow 0$. However, Bergin and Lipman (1996) also show that if the mutation probabilities in different states are allowed to go to zero at different rates, then any stationary distribution in the mutation-free process can be turned into the unique limiting distribution $\pi^* = \lim_{\epsilon \rightarrow 0} \pi^\epsilon$ of the process with mutations. They conclude: "In other words, any refinement effect from adding mutations is solely due to restrictions of how mutation rates vary across states." (Bergin and Lipman, p. 944).

What is missing in the modelling approaches in KMR and Young, and this is Bergin's and Lipman's main message, is a theory of why and how mutations occur. One possibility, suggested by Bergin and Lipman, is that mutation rates might be lower in high-payoff states than in low-payoff states, which might be expected if mutations are due to individuals' experimentation (see Bergin and Lipman, pp. 944,

945 and 947). Another reason why mutation probabilities may differ across population states, also suggested by Bergin and Lipman (pp. 945 and 955), is that mutations leading to larger payoff losses might have lower probabilities than mutations leading to smaller payoff losses. Bergin and Lipman do not investigate the consequences of either of these two ideas. We now follow up on their latter suggestion and show that it leads to a confirmation of the results of Young.

4.3. Endogenous mutation rates in Young's model. Let $h \in H$ denote the state of the population process in Young (1993). The state determines a range of possible probabilistic beliefs $\mu \in \Delta$ for each of the individuals drawn to play the game. More exactly, μ is one of those finitely many empirical statistically independent frequency distributions over the pure strategy sets which correspond to a sample of size k from the past m plays of the game. Since the state space H is finite, the set $\Sigma = \{\mu_{i-1} \in \Delta\}$ of possible beliefs μ is finite. Now suppose that, instead of an exogenous mistake probability ϵ , we take as exogenous the "disutility weight" \pm , a regular profile $\psi = (\psi_1, \dots, \psi_n)$ of disutility functions, and, for each state h in the finite state space H , an interior strategy profile $\sigma(h) = \sigma(\epsilon, h)$, the conditional (and potentially state dependent) error distribution in Young's (1993) model. Instead of taking ϵ to zero, we let \pm go to zero.

Since all mistake probabilities $\mu_i(\pm; \epsilon)$ are positive, for any $\pm > 0$, and the error

distribution $\hat{\pi}(h)$ is interior in all states h , the resulting stochastic process is ergodic, just as in Young's (1993) model, and thus has a unique invariant distribution π^\pm . We are interested in the limit as $\pm \rightarrow 0$, i.e. when the disutility of effort becomes insignificant in comparison with the game payoffs.

Proposition 3. If the disutility function profile is regular, then Theorems 2 and 3 in Young (1993), and Theorems 3.1 and 7.2 in Young (1998) hold. In particular, in a 2×2 coordination game with a unique risk-dominant equilibrium, this is the unique stochastically stable equilibrium.

Proof: This follows from proposition 2 above, combined with the proofs in Young (1993,1998). While Young establishes the result formally only for the case where all mistakes are a positive multiple of \pm (at each state h player i makes a mistake with probability \pm_i), it is easily verified that his arguments remain valid in the more general case considered here. To see this, note that if $!; !^0 \in \mathbb{R}^+$ and $i; j \in I$ are such that $\pi_i(!) \notin \Phi_i$ and $\pi_j(!^0) \notin \Phi_j$, then $l_i(!); l_j(!^0) > 0$ (since $\hat{\pi}(h)$ by hypothesis is interior for all $h \in H$). Proposition 2 implies that there for such $!; !^0 \in \mathbb{R}^+$ and $i; j \in I$ exist positive real numbers ϵ , δ and $\frac{1}{\pm}$, such that $\epsilon < \pi_i(\pm; !) = \pi_j(\pm; !^0) < \delta$ for all $\pm \in [\frac{1}{\pm}, \frac{1}{\pm}]$. The sets \mathbb{R}^+ and I being finite, we can find ϵ , δ and $\frac{1}{\pm}$ that work for all $!; !^0 \in \mathbb{R}^+$ and $i; j \in I$ such that $\pi_i(!) \notin \Phi_i$ and $\pi_j(!^0) \notin \Phi_j$. Hence, all mistake probabilities go to zero at the same rate, as $\pm \rightarrow 0$, in all states h which permit

no sample that renders some individual indifferent. In the excluded states, however, there is a positive probability that some mistake probability will optimally be chosen to be 1. That individual will in effect play his conditional error distribution, which assigns positive probability to all his pure strategies. But this is just as in Young's model, where all best replies are played with positive probability, which, in this case, means that all pure strategies are played with positive probability. The proofs of Theorems 2 and 3 in Young (1993) and Theorems 3.1 and 7.2 in Young (1998) thus apply. End of proof.

5. Counter-examples

5.1. **Non-similarity.** As indicated in section 3, it is easy to see that without a similarity assumption on the disutility functions, the results may depend on differences between disutility functions. For example, consider the following "battle of the sexes" type game:

	L	R
T	2; 1	0; 0
B	0; 0	1; 2

(13)

First, assume that only individuals in player population 2 make mistakes, and let their mistake probability be fixed and independent of the state. Then two thirds or more of the sample taken from population 2 has to be mistakes, in order to upset

(T; L), the equilibrium preferred by the error-free population 1. Similarly, it takes at least one third of mistakes to upset the other strict equilibrium. Hence, only the first equilibrium is stochastically stable. Second, note that this conclusion remains valid with a nice disutility function for each population, such that ϵ_1 is of smaller order than ϵ_2 .

5.2. Non-niceness. We now briefly turn to the case of disutility functions that are similar, in fact identical, but non-nice. In this case, Young's results need not hold. What might happen is that no stochastically stable equilibrium exists, since the sequence of invariant distributions π^\pm may fail to converge as $\epsilon \rightarrow 0$. The intuition is simple. Consider the following asymmetric payoff bi-matrix for a 2×2 coordination game:

	A	B
A	5; 1	0; 0
B	0; 0	2; 2

(14)

With p_i denoting the probability that player i assigns to his pure strategy A in the unique mixed equilibrium of this game, we have $p_1 = \frac{2}{3}$ and $p_2 = \frac{2}{7}$. Hence $p_1 + p_2 < 1$, so (A; A) is the risk-dominant equilibrium. With mistake probabilities that are of the same order of magnitude, we will, hence, select the equilibrium (A; A) in the limit.

However, a mistake by player 2 at this equilibrium incurs the smallest payoff loss in the two equilibria - this player loses only 1 payoff unit, while all other equilibrium payoff losses are at least 2 units. Consequently, the largest mistake probability occurs in state $(A; A)$. Now imagine a non-nice disutility function which is such that a mistake with a loss of only 1 is much more likely than a mistake resulting in a payoff loss of 2 or more. Then the limit outcome will be fully determined by the mistakes of player 2 at the state $(A; A)$. Obviously, if the only mistakes that count occur at this equilibrium, then the limit outcome will be $(B; B)$. Hence, for a non-nice disutility function, both $(B; B)$ and $(A; A)$ can be elements of the limit set.

In the remainder of this section, we formalize this observation. We first construct a non-nice disutility function.⁷ For this purpose, consider the sequence $\{h(x_n; y_n)\}_{n=1}^{\infty}$ in \mathbb{R}_+^2 defined by $x_1 = 1$, $y_1 = \frac{1}{3}$, and for all integers $n > 1$:

$$\begin{aligned}
 h(x_n; y_n) &= (2x_{n-1}; y_{n-1}^a) & \text{if } n \text{ is even} \\
 h(x_n; y_n) &= (3x_{n-1} + 2; y_{n-1}) & \text{if } n \text{ is odd}
 \end{aligned} \tag{15}$$

where $a > 1$. Clearly $\{x_n\}$ is an increasing sequence, going to plus infinity, and $\{y_n\}$ is a decreasing sequence, going to zero. Hence, there exist differentiable (to any order) and strictly decreasing functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $f(0) = \frac{1}{2}$, and $f(x_n) = y_n$ for all positive integers n . Any such function f decreases sufficiently on intervals $(x_n; x_{n+1})$ with n odd in order for $f(2x_n) = f(x_n) > 0$ to hold for n odd, as $n \rightarrow \infty$.

On the other intervals, however, f decreases sufficiently little in order for its total integral to be infinite (the integral of f over any interval $(x_n; x_{n+1})$ with n even is at least 1). The equation $v' = -f^{-1}$ defines a disutility function v that is non-nice, with f as the inverse to $-v'$.

For the bi-matrix game in (14), and with f as constructed above, we construct a sequence of ε 's going to zero, such that the associated sequence of stationary distributions in the limit places all probability mass on state $(B; B)$. To that end, for every odd integer n , let $\varepsilon_n = 1/x_n$. By definition of f , and condition (10)⁸,

$$\pi_2(\varepsilon_n; A) = y_n, \quad \pi_1(\varepsilon_n; B) = \pi_2(\varepsilon_n; B) = y_n^a, \quad \text{and} \quad \pi_1(\varepsilon_n; A) = y_n^a, \quad (16)$$

By following arguments as in Young (1993, Sect. 7) we can now show that the equilibrium $(A; A)$ is more easily upset than the equilibrium $(B; B)$ when $\varepsilon = \varepsilon_n$ with n odd. Let m be the common memory size of the players, let k be the (fixed) sample size and, let $(A; A)$ (resp. $(B; B)$) be the stationary state in which $(A; A)$ (resp. $(B; B)$) is played constantly. As in Young (1993), the path of least resistance from $(A; A)$ to $(B; B)$ either consists of player 1 making mistakes until player 2 has B as a best response to the sample of mistakes, or has player 2 making mistakes until player 1's best response is B . Let k^1 be the number of mistakes required by player 1 and let k^2 be the number required by player 2. Then

$$k^0 = [k=3] \text{ and } k^{00} = [5k=7] \quad (17)$$

where $[x]$ denotes the smallest integer equal to or larger than x . The probability to move from "(A; A)" to "(B; B)" is thus given by

$$\max f b_1(\pm; A)^{k^0}; b_2(\pm; A)^{k^{00}} g \quad (18)$$

If $\pm = \pm_n$ with n odd, and $a > 3$ then (16) implies that the easiest way to upset "(A; A)" is by player 2 making mistakes. Similarly, the easiest way to leave "(B; B)" is by player 2 making mistakes and the "resistance" of "(B; B)" is measured by

$$b_2(\pm; B)^{k^{00}} \text{ with } k^{00} = [2k=7] \quad (19)$$

Now, with $\pm = \pm_n$ and n odd, we have

$$\frac{b_2(\pm_n; B)^{k^{00}}}{b_2(\pm_n; A)^{k^{00}}} = \frac{y_n^{ak^{00}}}{y_n^{k^{00}}} = y_n^{(ak^{00} - k^{00})}; \quad (20)$$

so that, if $a > 3$, the ratio goes to zero as $n \rightarrow \infty$. Hence, for large n (and associated \pm_n) we have that "(A; A)" is much easier to upset than "(B; B)".

We conclude that for this sequence of \pm 's, converging to zero, the associated sub-

sequence of stationary distributions in the limit indeed places all probability mass on state $(B; B)$. At the same time, proposition 1 implies that there exists another sequence of ϵ 's, also going to zero, such that the associated subsequence of stationary distributions places all probability mass on state $(A; A)$ in the limit. Hence, the overall limit of stationary distributions does not exist for this disutility function.

Another way of formulating this observation is that, for every disutility function, the risk-dominant equilibrium belongs to the limit set of supports of stationary distributions, and that this limit set is a singleton if all disutility functions are nice.⁹

6. Conclusion

Bergin and Lipman (1996) showed that if mutation rates are state dependent, then the long-run equilibrium depends on exactly how these rates vary with the state. They conclude that the causes of mutations need to be modeled, in order to derive justifiable restrictions on how mutation rates depend on the state. In particular, they suggest that one might investigate the consequences of letting the probability of mistakes be related to the payoff losses resulting from these mistakes. This is exactly what we have done in this paper. We have developed a model in which mistakes are endogenously determined, and have shown that this model vindicates the original results obtained by Kandori, Mailath and Rob (1993) and Young (1993).

The model analyzed in this paper, although allowing for mistakes, is based on

strong rationality assumptions. Mistakes arise because players choose to make them, since it involves too much disutility to avoid them completely. Our individuals are hence unboundedly rational when it comes to decision making. Their lack of rationality is only procedural: At no disutility can they choose their own mistake probabilities in every population state. This is a very strong rationality assumption. However, we believe that our conclusions are robust in this respect, at least for generic 2×2 coordination games. For the effect of introducing control disutility was seen to only "deepen" the "basin of attraction" of the risk-dominant equilibrium, and hence speeding up the convergence to it. The limit result should therefore also be valid in intermediate cases of rationality. Such a robustness analysis would be highly relevant for the present approach.

While we restrict the analysis in this paper to the adaptive dynamics proposed in KMR and Young, the methodology can of course be applied also to other models which allow for mistakes in decision making, such as, to name just one example, Robson and Vega-Redondo (1996), where the long-run outcome in 2×2 coordination games is the Pareto dominant equilibrium, even if this is different from the risk dominant equilibrium. Hence, the present model is not an additional argument for risk dominance per se, but rather an argument for the robustness of a whole class of stochastic evolutionary models. As mentioned in the Introduction, Blume (1999) and Maruta (1997) derive general conditions under which the risk dominant equilibrium

is selected in $2 \in 2$ symmetric (one-population) games. It would be nice to have extensions of these results for more general games.

7. Appendix

7.1. Proof of proposition 1. Suppose that the claim does not hold. Since f_i is everywhere positive, we then have, for some $\epsilon > 0$:

$$\lim_{y \rightarrow 1} f_i(\epsilon, y) = f_i(y) = 0, \quad (21)$$

Thus $f_i(\epsilon, y) = f_i(y) - \frac{1}{2} \epsilon^{i-1}$ for all y sufficiently large, say $y > y_0$. Hence,

$$\int_{y_0}^1 f_i(y) dy = (\epsilon^{i-1} - 1) y_0 f_i(y_0) \times \int_{t=0}^1 2^{i-t} dt < 1, \quad (22)$$

and thus $\int_0^1 f_i(y) dy < 1$ since f_i is decreasing with $f_i(0) = 1$. But we also have

$$\int_0^1 f_i(y) dy = \int_0^1 v_i^0(p) dp = \lim_{t \rightarrow 0} v_i(t) - v_i(1) = +1, \quad (23)$$

a contradiction, proving the claim in the proposition.

7.2. Niceness. We next prove that if v_i is nice, then

$$\lim_{y \rightarrow 1} \inf_{\epsilon > 0} f_i(\epsilon, y) = f_i(y) > 0 \quad \text{for all } \epsilon > 0. \quad (24)$$

Assume that the condition in the definition of niceness holds for $\lambda_0 > 1$. Then it holds for all $\lambda \in (0; \lambda_0)$, by monotonicity of f_i . And if $\lambda > \lambda_0$, then $(\lambda_0)^n > \lambda$ for some positive integer n . Again by monotonicity of f_i :

$$\liminf_{y \downarrow 1} f_i(\lambda y) = f_i(y) \leq \liminf_{y \downarrow 1} f_i(\lambda_0^n y) = f_i(y) \quad (25)$$

$$= \liminf_{y \downarrow 1} \frac{f_i(\lambda_0^n y)}{f_i(\lambda_0^{n-1} y)} \leq \frac{f_i(\lambda_0^n y)}{f_i(\lambda_0^{n-1} y)} \leq \dots \leq \frac{f_i(\lambda_0 y)}{f_i(y)} \quad (26)$$

$$\leq \liminf_{y \downarrow 1} f_i(\lambda_0 y) = f_i(y) > 0 \quad (27)$$

7.3. Proof of Lemma 1. Under the hypothesis of the lemma, the function λv_i^0 is convex. Thus, for any $y > 0$, $\lambda > 1$, and $\mu \in (0; 1)$ such that $y = \lambda v_i^0(\mu)$:

$$\frac{\lambda y - y}{f_i(y) - f_i(\lambda y)} \leq v_i^0(\mu). \quad (28)$$

(To see this, draw a diagram with μ on the horizontal axis, and $\lambda v_i^0(\mu)$ on the vertical axis.) Equivalently,

$$f_i(\lambda y) \leq f_i(y) - (\lambda - 1) \frac{y}{v_i^0(\mu)}, \quad (29)$$

or

$$\frac{f_i(\zeta_i y)}{f_i(y)} \leq 1 + (\zeta_i - 1) \frac{v_i^0(i)}{v_i^{00}(i)}.$$

Consequently,

$$\liminf_{y \rightarrow 1} \frac{f_i(\zeta_i y)}{f_i(y)} \leq 1 + (\zeta_i - 1) \liminf_{i \rightarrow 0} \frac{v_i^0(i)}{v_i^{00}(i)}. \quad (30)$$

But under the last hypothesis of the lemma, there exists a scalar $\zeta_i > 1$ such that

$$\limsup_{i \rightarrow 0} \frac{i v_i^0(i)}{v_i^{00}(i)} < \frac{1}{\zeta_i - 1}. \quad (31)$$

Hence, for this ζ_i ,

$$1 + (\zeta_i - 1) \liminf_{i \rightarrow 0} \frac{v_i^0(i)}{v_i^{00}(i)} > 0. \quad (32)$$

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Notes

¹As a by-product, we obtain a new refinement of the Nash equilibrium concept similar to Selten's (1975) notion of "trembling hand" perfect equilibrium. Whereas in Selten's model both the mistake probabilities and the conditional error distribution in case of a mistake are exogenous, the mistake probabilities are here endogenous.

²Since $\bar{\pi}_i$ is interior, this implies that π_i is undominated.

³Since $\bar{\pi}_i$ is interior, this happens if and only if all pure strategies available to player i are best replies to $\bar{\pi}_i$.

⁴The relative risk aversion is usually defined for utility, rather than disutility, functions. Note, however, that $\rho_i^0(\cdot) = \rho_i^0(\cdot)$ is the relative risk aversion of the (sub)utility function v_i . This parallel is particularly clear in case $\lim_{\epsilon \rightarrow 0} v_i(0)$ is finite, since then $\rho_i^0(\cdot) = \rho_i^0(\cdot)$ is also the relative risk aversion of the normalized (sub)utility function $U_i : (0, 1] \rightarrow \mathbb{R}_+$ defined by $U_i(\cdot) = v_i(0) - v_i(\cdot)$, an increasing concave function with $U_i(0) = 0$.

⁵Similar results are obtained in Blume (1999) for symmetric 2x2-coordination games. In Blume's model, individuals revise their strategy choice probabilistically as a function of the current payoff difference between the two pure strategies. Our model is complementary to Blume's in deriving probabilistic choice behavior from a decision-theoretic model, and differs from Blume's by inducing choice probabilities that not only depend on payoff differences but also on potential payoff losses (from own mistakes).

⁶A curb set is a product set that is closed under rational behavior, i.e., that contains all its best replies, see Basu and Weibull (1991).

⁷The following construction of a function f , that is the negative of the inverse of a non-nice disutility function, is an adaptation of a function suggested to us by Henk Norde, Tilburg University.

⁸To see this, note that if $x_n = 1 - x_n$ then $b_1(x_n; \cdot) = f(1 - x_n)$. Hence, $b_2(x_n; A) = f(x_n) = y_n$, and $b_1(x_n; B) = b_2(x_n; B) = f(2x_n) = f(x_{n+1}) = y_n^a$.

⁹By the limit set of supports of stationary distributions, for a given control-cost function, we here mean the union of the supports of probability distributions π^t , each of which is the limit to some subsequence of stationary distributions (i.e., associated with some sequence of positive ϵ^t 's converging to zero).